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SADDLEPOINT APPROXIMATION TO THE CUMULATIVE DISTRIBUTION FUNCTION FOR A LINEAR COMBINATION OF CONVOLUTION GAMMA–EXPONENTIAL MODELS

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ABSTRACT

This paper derives the saddlepoint approximation for a linear combination of the convolution $aX + bY$, where $a > 0$ and $b > 0$ are real constants and X and Y denote Gamma and Exponential random variables, respectively, and are distributed independently of each other. We also discuss the generalization of this convolution (i.e., the convolution of the convolution for Gamma-Exponential random variables), which is unknown and extremely difficult to obtain. The associated saddlepoint approximation CDFs are derived, and plots of the CDFs and some measures of the center and spread of the new approximations are given.

Keywords: Gamma-Exponential distribution, Saddlepoint approximation, moment generating function, linear combinations of convolutions.

INTRODUCTION

Distributions consisting of a linear combination of independent random variables are commonly found in different areas of research (Ladekarl *et al*., 1997; Amari and Misra, 1997; Cigizoglu and Bayazit, 2000). Many researchers have studied this type of distribution while considering X and Y as random independent variables from the same family. For instance, see Fisher (1935) and Chapman (1950) for the Student's t family, Christopeit and Helmes (1979) for the normal family, Davies (1980) and Farebrother (1984) for the chi-squared family, Ali (1982) for the exponential family, Moschopoulos (1985) for the Gamma family, Dobson *et al*. (1991) for the Poisson family, and Witkovsky (2001) for the inverted Gamma family. For the case in which X and Y belong to different families, it is obvious that sufficient statistical research is lacking. One important case in which the method of approximation must be used arises when the exact CDF is unknown or is difficult to compute.

This paper provides a saddlepoint approximation for linear combinations of convolution Gamma-Exponential models in which the expression for the CDF is complicated. The exact "true" density, computed from a numerical convolution of the densities is involved. The complicated exact expression follows a particular function, such as the incomplete Gamma, complementary incomplete Gamma, or error function (Stewart *et al*., 2007). This special function may lead to intractable computations. However, in some cases, numerical computation suggests that the function is incorrect because the result differs substantially from the numerical convolution (Butler, 2007).

We extend the convolution approach to find the convolution of the convolution Gamma-Exponential, which is more complicated and for which the exact CDF is not yet known.

The organization of this paper is as follows. In Section 1, the fundamentals of linear combinations of random variables and some basic points regarding the saddlepoint approximations are presented. The approximation expressions for the CDFs are derived in Section 4. The resulting plots are provided, and some measures of the center and spread are given as examples in Section 5.

1. Linear Combination of Convolution Gamma–Exponential Distributions

The Gamma-Exponential distribution is a convolution of

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a Gamma and an Exponential distribution. There are various practical applications of this distribution in different scientific areas, including control, automation, and fuzzy sets. Let $Z = aX + bY$, In this equation, $a > 0$ and $b > 0$ are real constants and the variables X and Y denote Gamma and Exponential random variables that are distributed independently. The derivation of the distribution of the linear combination can be considered by using the saddlepoint approximation method. The Gamma and Exponential random variables are defined as follows.

1.1 Gamma Distribution: A continuous random variable X is said to have a Gamma distribution if it's PDF $F(x) = P(X \leq x)$ and CDF are, respectively given by

$$f(x) = \frac{1}{\theta^k \Gamma(k)} x^{k-1} e^{-\frac{x}{\theta}} \quad , x > 0 \text{ and } k, \theta > 0$$

Here $\Gamma(k)$ is the Gamma function evaluated at k.

$$F(x) = \frac{1}{\Gamma(k)} \gamma\left(k, \frac{x}{\theta}\right),$$

Where $\gamma\left(k, \frac{x}{\theta}\right)$ is the lower incomplete Gamma function.

With mean and variance are, respectively given by

$$E(x) = k\theta, \quad V(x) = k\theta^2,$$

The MGF of the Gamma distribution is given by

$$M_x(s) = (1 - \theta s)^{-k},$$

1.2 Exponential Distribution: A continuous random variable y is said to have an exponential distribution if its PDF $f(y)$ is given by

$$f(y) = \frac{1}{\alpha} e^{-\frac{y}{\alpha}} \quad , \quad y > 0 \text{ and } \alpha > 0$$

and its CDF $F(y) = P(Y \leq y)$ can be expressed as

$$F(y) = 1 - e^{-\frac{y}{\alpha}} \quad ,$$

The mean and variance, respectively are given by

$$E(y) = \alpha \,, \quad V(y) = \alpha^2,$$

The MGF of the exponential distribution is given by

$$M_y(s) = (1 - \theta s)^{-1},$$

2. Properties of a Linear Combination of Convolution Gamma-Exponential Distributions:

The distribution of a linear combination of random variables emerges in different practical issues, and thus, several researchers have studied this area. This section provides a derivation of the distribution of a linear combination of random variables while considering the properties of this distribution. The mathematical form of this linear combination can be provided as follows:

$$S_N = c_1 X_1 + c_2 X_2 + c_3 X_3 + \cdots + c_N X_N,$$

In the above equation, the coefficients c_1 through c_N are multiplied by corresponding variables X_1 through X_N.

The mean and variance, respectively are given by

$$E(S_N) = \sum_{i=1}^{N} c_i E(X_i), \quad V(S_N) = \sum_{i=1}^{N} c_i^2 V(X_i)$$

The saddlepoint approximations are based on the MGF which is given by

$$M_{S_N}(s) = \prod_{i=1}^{N} M_{X_i}(c_i s)$$

Where $M_{X_i}(.)$ is the MGF for the random variable X_i , $i = 1,2, \ldots, N$.

3. Saddlepoint Approximation & Linear Combinations of Random Variables:

The most fundamental saddlepoint approximation was introduced by Daniels (1954) and is essentially a formula for approximating a density or mass function from its associated MGF. This method is executed by performing various operations on the MGF or, equivalently, the cumulant generating function (CGF), of a random variable. For the development and discussion of saddlepoint methodology and related methods, see Daniels (1954) and (1987) for details of the density and mass approximation, Skovgaard (1987) for a conditional version of this approximation, Reid (1998) for applications to inference, Borowiak (1999) for a discussion of a tail area approximation with uniform relative error, and Terrell (2003) for a stabilized Lugannani-Rice formula.

3.1 Saddlepoint Approximation for Univariate Density and Mass Function:

Let X be a continuous or discrete random variable with (density or mass) $f(x)$ and with MGF $M(s)$ and CGF $K(s)$. The saddlepoint approximation for $f(x)$ is given by

$$\hat{f}(x) = \frac{1}{\sqrt{2\pi K''(\hat{s})}} exp\{K(\hat{s}) - \hat{s}x\}$$

Where $\hat{s} = \hat{s}(x)$ is the unique solution to the saddlepoint equation $K'(\hat{s}) = x$. This method is meaningful for values of X that are interior points of $\{x: f(x) > 0\} = \chi$.

3.2 Saddlepoint Approximation for Univariate Continuous Distribution Functions

Suppose a continuous random variable X has CDF F and CGF K with mean μ. The saddlepoint approximation for $F(x)$, as introduced by Lugannani and Rice (1980), is

$$\hat{F}(x) = \begin{cases} \Phi(\hat{w}) + \phi(\hat{w})(1/\hat{w} - 1/\hat{u}) & if x \neq \mu \\ \frac{1}{2} + \frac{K'''(0)}{6\sqrt{2\pi K''(0)^{3/2}}} & if x = \mu \end{cases}$$

Where $\emptyset$ and Φ denote the standard normal density and the CDF, respectively, and

$$\hat{w} = sgn(\hat{s})\sqrt{2\{\hat{s}x - K(\hat{s})\}}, \quad \hat{u} = \hat{s}\sqrt{K''(\hat{s})}$$

are functions of x and the saddlepoint $\hat{s}$ and $sgn(\hat{s})$ captures the *sign* ± for $\hat{s}$.

4. Saddlepoint Approximation for of a Linear Combination of Convolution:

Saddlepoint Approximation for of a Linear Combination of Convolution can defined as:

4.1 Convolution Gamma-Exponential Distribution:

The linear combination of this distribution is

$$S_N = c_1X_1 + c_2X_2$$

Where

$X_1 \sim$ Gamma (θ,k) , $X_2 \sim$ Exponential (α)

$X_i \geq 0$, and c_i are real constants where $i = 1,2$.

The MGF for a linear combination of Gamma-Exponential distributions is

$$M_{S_N}(s) = \prod_{i=1}^{N} M_{X_i}(c_i s) = (1 - \theta c_1 s)^{-k}(1 - \alpha c_2 s)^{-1},$$

$$i = 1,2$$

Then, the CGF is

$$K_{S_N}(s) = -k ln(1 - \theta c_1 s) - ln(1 - \alpha c_2 s)$$

and the saddlepoint equation $K'_{S_N}(s) = x$ becomes

$$K'_{S_N}(\hat{s}) = \frac{k\theta c_1}{1 - \theta c_1\hat{s}} + \frac{\alpha c_2}{1 - \alpha c_2\hat{s}} = x$$

The saddlepoint solution is a root of a saddlepoint equation and determined numerically.

The saddlepoint technique also depends on the second and third derivatives of the saddlepoint equation which are respectively given by

$$K''_{S_N}(\hat{s}) = \frac{k(\theta c_1)^2}{(1 - \theta c_1\hat{s})^2} + \frac{(\alpha c_2)^2}{(1 - \alpha c_2\hat{s})^2}$$

$$K'''_{S_N}(\hat{s}) = \frac{2k(\theta c_1)^3}{(1 - \theta c_1\hat{s})^3} + \frac{2(\alpha c_2)^3}{(1 - \alpha c_2\hat{s})^3}$$

Based on above calculation we find

$(1 - \theta c_1\hat{s}) \neq 0$ and $(1 - \alpha c_2\hat{s}) \neq 0$

This result leads to

$$\hat{s} \neq \frac{1}{\theta c_1} \quad and\ \hat{s} \neq \frac{1}{\alpha c_2}$$

Furthermore

$(1 - \theta c_1\hat{s}) > 0$ and $(1 - \alpha c_2\hat{s}) > 0$, then we get

$$\hat{s} < \frac{1}{\theta c_1} \quad and \quad \hat{s} < \frac{1}{\alpha c_2}$$

So, this implies

$$\hat{s} < \min\left(\frac{1}{\theta c_1}, \frac{1}{\alpha c_2}\right)$$

Thus, we can obtain the saddlepoint approximation to the cumulative distribution function for a linear combination of Gamma-Exponential distributions, as given above.

4.2 Generalized Convolution Gamma–Exponential Distribution:

The linear combination of this convolution is

$$L_N = c_1X_1 + c_2X_2 + c_3X_3 + \cdots + c_NX_N$$

where $X_i's$ are i.i.d. random variables, $X_i's \geq 0$, and $X_i's \sim$ convolution Gamma-Exponential $(0,k,\alpha)$, c_1, $1=1,2,\ldots,N$ are real constants.

The MGF for the convolution L_N is the product of the MGFs for the N random variables, i.e.

$$M_{L_N}(s) = \prod_{i=1}^{N} M_{c_iX_i}(s)$$

$$= \prod_{i=1}^{N} (1 - \theta c_i s)^{-k}(1 - \alpha c_i s)^{-1}$$

and the CGF is expressed as

$$K_{L_N}(s) = ln \prod_{i=1}^{N} (1 - \theta c_i s)^{-k}(1 - \alpha c_i s)^{-1}$$

$$= \sum_{i=1}^{N} ln\ (1 - \theta c_i s)^{-k}(1 - \alpha c_i s)^{-1}$$

$$= \sum_{i=1}^{N} [ln\ (1 - \theta c_i s)^{-k} + ln(1 - \alpha c_i s)^{-1}]$$

The first derivative is

$$K'_{L_N}(\hat{s}) = \sum_{i=1}^{N} [ln\ (1 - \theta c_i\hat{s})^{-k} + ln(1 - \alpha c_i\hat{s})^{-1}]$$

Above calculations is referred to as the saddlepoint equation, $\hat{s}$ is the saddlepoint associated with value $\mathbf{x}$ and the saddlepoint is the unique solution to $K'_{L_N}(\hat{s}) = X$, which can be solved using a numerical method, for example; a Newton Raphson method.

The second and third derivatives are

$$K''_{L_N}(\hat{s}) = \sum_{i=1}^{N} \frac{k(\theta c_i)^2}{(1-\ \theta c_i \hat{s}\ )^2} + \sum_{i=1}^{N} \frac{(\alpha c_i)^2}{(1-\ \alpha c_i \hat{s}\ )^2}$$

$$K'''_{L_N}(\hat{s}) = \sum_{i=1}^{N} \frac{2k(\theta c_i)^3}{(1-\theta c_i \hat{s}\ )^3} + \sum_{i=1}^{N} \frac{2(\alpha c_i)^3}{(1-\alpha c_i \hat{s}\ )^3}$$

As in previous calculations from the above expressions, it is clear that

$$\hat{s} < \min\left(\frac{1}{\theta c_i}, \frac{1}{\alpha c_i}\right) \, and \, \hat{s} \neq \frac{1}{\theta c_i}, \hat{s} \neq \frac{1}{\alpha c_i}$$

Then, we can derive the saddlepoint approximations based on the above functions to obtain the PDF and CDF.

5. Applications to Saddlepoint Approximation for Linear Combination of Convolution Gamma-Exponential Distribution

The distribution of linear combinations of the form $aX + bY$ is found to be an interesting area of study in the problems related to the automation, control, fuzzy sets, and neurocomputing. Some of the aspects of its implementation are provided as follows:

i. Considering automation control, it can be said that the issue of maximizing the expected sum of n variables are found to be common, which are selected from a sequence of N sequentially acquiring the independently and identically distributed scalar random variables ranging from $X_1, X_2, \ldots, X_N$. It provides the objective of devising a decision rule for maximizing the $\sum_{i=1}^{N} X_{k_i}$, where X_{k_i} is the index of the ith selected random variable. The random variable X_k is observed at time k and the decision provides the non-selection or selection of value is required to be taken online.

ii. In computer science, the theory of congruence equations has many applications and there is vast literature regarding the congruence equations. The derivation of interesting formulas and functions is observed in the past two decades, which include the expressions giving different solutions of linear congruence. It is found that these solutions have certain relationships with the statistical issues such as the distribution of the values of specific sums.

Example 1:

(Linear combination of convolution Gamma-Exponential distribution)

$$S_N = c_1 X_1 + c_2 X_2$$

where

$X_1 \sim Gamma(0.5,1),\ X_2\ Exponential\ (2)$ and $c_1 = 1,\ c_2 = 2$ when $x = 1$ From the previous equations in this paper, the saddlepoint is $K'_{S_N}(\hat{s}) = 1.$ then, we get

$$\frac{k\theta c_1}{1-\ \theta c_1 \hat{s}} + \frac{\alpha c_2}{1-\ \alpha c_2 \hat{s}} = X$$

Then we can obtain the saddlepoint as

Let

$a = x\alpha\theta c_1 c_2\,, \quad b = \alpha\theta c_1 c_2 k + \alpha\theta c_1 c_2 - x\theta c_1 - x\alpha c_2,$

$c = x - \alpha c_2 - k\theta c_1$

by solving the saddlepoint equation we can derive the saddlepoint as

$$\hat{s} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where

$\alpha = 2, \quad \theta = 1, \quad c_1 = 1, \quad c_2 = 2, \quad k = 0.5$

so,

$$\hat{s} = \frac{1 - \mp 7.5498}{8}$$

Based on the previous Equations, we get $\hat{s} < 0.25$, then we take

$$\hat{s} = \ -1.068625$$

and

$$\hat{w} = \ -\sqrt{2(-1.068625\ (1) + 2.0256)} = -1.383455$$

also, we can calculate $\hat{u}$ as:

$$\hat{u} = -1.068625\ \sqrt{(0.6925)} = -0.8888$$

where

$$K''_{S_N}(\hat{s}) = 0.6925$$

Then saddlepoint approximation to cumulative distribution function (CDF) for the linear combination of Gamma–Exponential distribution is given by

$$\hat{F}(1) \ = \Phi(-1.384\ ) + \phi(-1.384\ )(-1/1.384\ \ +1/0.88895)$$

That implies

$$\hat{F}(1) = 0.1447$$

We used MATLAB program to obtain various CDFs approximations for any value of $X > 0$. On the other hand, to find the exact distribution and we used this program to simulate one million observations for each of the x_1, x_2. Then, we calculate the linear combination of the Gamma-Exponential variable as:

$$S_{N_{c_i}} \ = \ x_{1i} + x_{2i} \quad , \qquad i = 1,2, \ldots, 10^6$$

and the exact distribution is approximated using

$$F_N(x) = \frac{1}{N}\sum_{i=1}^{N} I(S_{N_{c_i}} \leq x)$$

However, we can obtain important statistical measures from a cumulative frequency distribution. These are useful when we want to compare two or more data sets. The inter quartile range (IQR) yields an improved measure of the spread of the data and is given by the following:

IQR = upper quartile - lower quartile

Using MATLAB program we get

$$IQR = \hat{F}^{-1}(0.75) - \hat{F}^{-1}(0.25) = 6.11 - 1.6 = 4.51$$

As well as, we can derive some measures of central tendency as the median which is given by

$$\hat{F}^{-1}(0.5) = 3.3$$

Example 2:

(Linear combination of generalized convolution Gamma-Exponential distribution)

The linear combination of this convolution is

$$L_N = c_1X_1 + c_2X_2 + c_3X_3 + \cdots + c_NX_N$$

Where X_i's are i.i.d random variable, $X \geq 0$,

$$X_i \sim convolution\ Gamma - Exponential\ (\theta, k, \alpha),\quad c_i\ , i = 1,2,\dots,N$$

Let

$\theta = 2,\quad k = 0.6,\quad \alpha = 3$

and $c_1 = 2,\ c_2 = 3,\ c_3 = 1,\ c_4 = 2$

$$N = 4,\quad \sum_{i=1}^{4} c_i = 2 + 3 + 1 + 2 = 8,$$

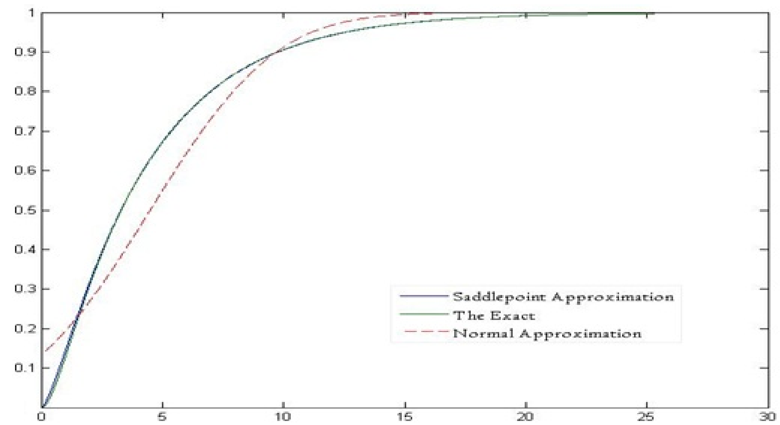

Fig. 1. Comparison saddlepoint approximation and the exact with normal approximation for linear combination of Gamma-Exponential (0.5,1,2).

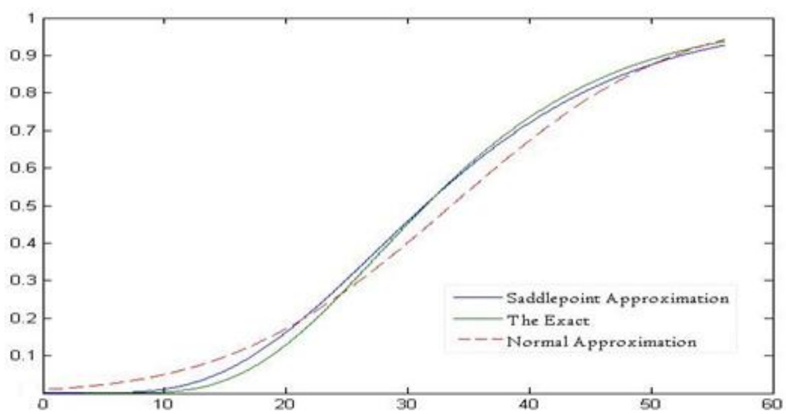

Fig. 2. Comparison saddlepoint approximation and the exact with normal approximation for linear combination of Convolution Gamma–Exponential (2,0.6,3).

$$\sum_{i=1}^{4} c_i^{\,2} = 4 + 9 + 1 + 4 = 18$$

With first derivative for CGF

$$K'_{L_N}(\hat{s}) = \frac{k\theta c_1}{1-\theta c_1\hat{s}} + \frac{k\theta c_2}{1-\theta c_2\hat{s}} + \frac{k\theta c_3}{1-\theta c_3\hat{s}} + \frac{k\theta c_4}{1-\theta c_4\hat{s}} + \frac{\alpha c_1}{1-\alpha c_1\hat{s}} + \frac{\alpha c_2}{1-\alpha c_2\hat{s}} + \frac{\alpha c_3}{1-\alpha c_3\hat{s}} + \frac{\alpha c_4}{1-\alpha c_4\hat{s}}$$

The saddlepoint equation $K'_{L_N}(\hat{s}) = X$ is given by

$$\frac{2.4}{1-4\hat{s}} + \frac{3.6}{1-6\hat{s}} + \frac{1.2}{1-2\hat{s}} + \frac{2.4}{1-4\hat{s}} + \frac{6}{1-6\hat{s}} + \frac{9}{1-9\hat{s}} + \frac{3}{1-3\hat{s}} + \frac{6}{1-6\hat{s}} - X = 0$$

This can be solved using a numerical method for example Newton Raphson method. Then, we get the saddlepoint with initial value $\hat{s}_0 = 0$ as

$$\hat{s} = -0.1636$$

For $i = 1,2,3,4$ we get

$$\frac{1}{\theta c_i} = \frac{1}{4}, \frac{1}{6}, \frac{1}{2}, \frac{1}{4}$$

$$\frac{1}{\alpha c_i} = \frac{1}{6}, \frac{1}{9}, \frac{1}{3}, \frac{1}{6},$$

and the saddlepoint achieves $\hat{s} < \frac{1}{9} = 0.1111$

using the saddlepoint $\hat{s}$ above when, $X = 0.02$ we get

$$\hat{w} = sgn(\hat{s})\sqrt{2[\hat{s}x - K(\hat{s})]} = -\sqrt{2[(-0.1636)(0.02) + 3.8566)]} = -2.77608$$

$$\hat{u} = \hat{s}\sqrt{K''(\hat{s})} = -0.1636\sqrt{13.8782} = 0.609466$$

where $K(\hat{s}) = -3.8566$ and $K''(\hat{s}) = 13.8782$

The saddlepoint approximations to the CDF for this example is

$$\hat{F}(0.02) = \Phi(-2.7760) + \phi(-2.7760)(-1/2.7760 + 1/0.609466)$$

So, we get

$$\hat{F}(0.02) = 0.01363$$

We used MATLAB program to find the saddlepoint CDFs for any value of $x > 0$. The empirical distribution is used to determine the exact cumulative distribution function, we simulate 10^6 Independent values of L_N for each of X_1, X_2, X_3 and X_4.

Then calculate the linear combination as:

$$L_{Nc_i} = 2X_{1i} + 3X_{2i} + X_{3i} + 2X_{4i},$$

$$i = 1,2,\ldots,10^6$$

As a result of that, the exact distribution is approximated using

$$F_{Nc_i}(x) = \frac{1}{N}\sum_{i=1}^{N} I(L_{Nc_i} \le x)$$

CONCLUSION

Saddlepoint techniques are powerful tools for extracting accurate approximations to densities and distribution functions of the sums of continuous random variables. The Lugannani-Rice formula is one of the simplest forms of the saddlepoint approximation. The construction of the saddlepoint approximation and a simulation method for approximating the CDF for linear combinations of convolution Gamma-Exponential distributions are presented in this paper. Based on a comparison of the CDF obtained from the saddlepoint approximation and that of the exact, the different values of the saddlepoint approximations are similar to the exact value for both distributions. It is apparent that the two approximations are the same. The computations were carried out in MATLAB, and the accuracy of the saddlepoint computations results from two factors: the intrinsic accuracy of the computational routines in the software and the number of digits being carried in the calculations to neutralize the round-off error.

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